Stochastic processes with finite correlation time: Modeling and application to the generalized Langevin equation

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The kangaroo process (KP) is characterized by various forms of covariance and can serve as a useful model of random noises. We discuss properties of that process for the exponential, stretched exponential, and algebraic (power-law) covariances. Then we apply the KP as a model of noise in the generalized Langevin equation and simulate solutions by a Monte Carlo method. Some results appear to be incompatible with requirements of the fluctuation-dissipation theorem because probability distributions change when the process is inserted into the equation. We demonstrate how one can construct a model of noise free of that difficulty. This form of the KP is especially suitable for physical applications.

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I. INTRODUCTION

It is frequently assumed that a noise in stochastic equations is not correlated and the underlying stochastic process can be regarded as Markovian. This assumption is justified only if the time scales involved are large compared to the noise correlation time. On the other hand, the noise itself can result from a procedure of fast modes removal. It is well known [1,2] that in such cases noise correlations, both in space and time, must arise. For some stochastic processes, the noise covariance decays fast with time and it can be put in an exponential form (the colored noise [3]). Frequently even longer tails of the noise covariance are observed. Algebraic covariances appear in the fluid dynamics [4–6] and linearized hydrodynamics [7]; they are responsible for such phenomena as noise-induced Stark broadening [8] and anomalous nuclear scattering in the framework of the molecular dynamics [9]. A direct consequence of the algebraic form of the velocity autocorrelation function, falling not faster then $1/t$, is the infinite value of the diffusion coefficient [10]. An anomalous diffusion process is frequently observed in disordered media where a trapping mechanism leads to algebraic distributions of waiting time. For example, some amorphous insulating materials (e.g., As_2Se_3) exhibit a highly dispersive transient photocurrent [11], just due to charge hopping between spatially disordered sites.

Stochastic dynamics driven by a noise different from the white noise obeys the generalized Langevin equation (GLE) [12,13]:

$$m\frac{dv(t)}{dt} = -m\int_0^t K(t-\tau)v(\tau)d\tau + F(t), \tag{1}$$

where $F(t)$ is a stochastic force and m denotes the mass of the Brownian particle. Due to the second fluctuation-dissipation theorem (FDT) [14,15], the kernel $K(t)$ can be expressed in terms of the noise covariance $C_F \equiv \langle F(0)F(t)\rangle$: $K(t)=C_F/mT$, with the temperature T. The Eq. (1) can be handled as a usual Volterra equation [16]. Assuming the initial condition $v(0)=0$, the general solution can be expressed in the form of a stochastic integral [17]:

$$v(t) = m^{-1}\int_0^t R(t-\tau)F(\tau)d\tau, \tag{2}$$

where the Laplace transform of the resolvent $R(t)$ is given by the equation

$$\tilde{R}(s) = 1/[s+\tilde{K}(s)]. \tag{3}$$

From Eq. (2), expressions for some average quantities follow. For the velocity variance, we have

$$\langle v^2\rangle_S(t) = m^{-2}\int_0^t\int_0^t R(t-\tau)R(t-\tau')C_F(|\tau-\tau'|)d\tau d\tau', \tag{4}$$

where the average $\langle\ \rangle_S$ is taken over an equilibrium ensemble with some stationary probability distribution. The FDT ensures that asymptotically, for large time, the system reaches the equilibrium value $\langle v^2\rangle_S = T/m$ (the equipartition energy rule). For the velocity autocorrelation function, in turn, we have simply

$$C_v(t) \equiv \langle v(0)v(t)\rangle_S = (T/m)R(t). \tag{5}$$

The assumption about a form of noise covariance is sufficient to calculate some average quantities. In order to simulate stochastic trajectories from Eq. (2), one needs a concrete physical process that could serve as a model of the noise. For example, for the exponential covariance, it could be the well-known Ornstein-Uhlenbeck process, which, due to the Doob theorem [18], is very important if amplitude distributions are Gaussian. A broad class of stochastic processes known as ''kangaroo processes'' (KP) [19] is especially interesting. One can construct the KP for an arbitrary, given form of covariance. The KP is particularly well suited for problems involving algebraic, scale-invariant dependences. Long tails of constant value of the process in the step-wise structure of the KP make possible to preserve the memory about this value for a sufficiently long time to produce such slowly decaying form of the covariance. Due to that structure, the KP resembles stochastic, dispersive transport processes in disordered media, e.g., the hopping time distribution

[20–22]. In the framework of random walk processes, a pattern of long straight-line segments is typical for Lévy flights [23–25].

This paper deals with random noises possessing various covariances: exponential, stretched exponential and algebraic, and expresses them in terms of the KP. The most important properties of KP are summarized in Sec. II; we also derive there formulas referring to those forms of the covariance. In Sec. III, we consider the application of the KP as a model of random force in the GLE. The most important results are summarized and discussed in Sec. IV.

II. THE KANGAROO PROCESS

The kangaroo process [19] is a stepwise, discontinued random function. The value of the process $m(t)$ is determined at subsequent random jumping times $t_1, t_2, \ldots$. The jumping frequency $\nu(m)$ depends on the value of the process itself and m remains constant between jumps. We introduce also the interval length as a reciprocal of the frequency: $s=1/\nu$. Due to some physical applications, this quantity can also be called ''a free path.'' The KP is a stationary Markov process and can be defined by the probability density $p(m,t)$ satisfying the following Fokker-Planck equation

$$\frac{\partial}{\partial t}p(m,t)=\nu(m)\left(-p(m,t)+\frac{P_{KP}(m)}{\int \nu(m')P_{KP}(m')dm'}\times\int \nu(m')p(m',t)dm'\right), \tag{6}$$

where $P_{KP}(m)$ denotes a stationary probability distribution of $m(t)$. The interval length s is also a stochastic quantity. Its probability distribution $P(s)$ is connected with $P_{KP}(m)$ by the relation $P(s)ds=2P_{KP}(|m|)d|m|$. We assume that $P_{KP}(m)$ and $\nu(m)$ are even functions of m. This assumption allows us to get a simple expression for the covariance of KP $C(t)=\langle m(t)m(0)\rangle_S$ where the average is taken over the stationary probability distribution $P_{KP}(m)$:

$$C(t)=\int_{-\infty}^{+\infty} m^2 P_{KP}(m)\exp[-\nu(m)|t|]dm. \tag{7}$$

We want to derive an expression for $\nu(m)$ for a given covariance $C(t)$ and an amplitude distribution $P_{KP}(m)$. Let us assume that $\nu(m)$ is a monotonic increasing function of $|m|$ and $\nu(\infty)=\infty$. Then we can change the integration variable in Eq. (7) and obtain the Laplace integral

$$C(t)=2\int_{\nu(0)}^{+\infty} m^2 P_{KP}(m)\frac{dm}{d\nu}\exp(-\nu|t|)d\nu. \tag{8}$$

Therefore, $\nu(m)$ is a solution of the following differential equation

$$\frac{d\nu}{dm}=2m^2 P_{KP}(m)/\tilde{C}(\nu), \tag{9}$$

where $\tilde{C}(\nu)$ denotes the inverted Laplace transform of $C(t)$.

Solution of the Eq. (9) allows us, in principle, to generate a stochastic time series of the process with a given, quite arbitrary covariance and with an arbitrary distribution $P_{KP}(m)$.

A. Exponential covariance

The KP for the exponential covariance

$$C(t)=\nu_0\exp(-\nu_0 t), \tag{10}$$

where $\nu_0=$const is a reciprocal of the correlation time, is called the Kubo-Anderson process [26]. The jumping times are uniformly distributed in the interval $(-\infty,\infty)$ with a m-independent density ν_0, according to the Poissonian distribution. Therefore, the intervals of constant m are distributed exponentially: $P(s)=\nu_0\exp(-\nu_0 s)$. The value of the process $m(t)$ may be chosen according to an arbitrary distribution $P_{KP}(m)$. In that sense, the distributions of m and s are independent of each other. That property holds only for the Kubo-Anderson process; for a general KP m and s are interdependent. The Fokker-Planck equation (6) takes a simpler form for the Kubo-Anderson process

$$\frac{\partial}{\partial t}p(m,t)=\nu_0\left(-p(m,t)+P_{KP}(m)\int p(m',t)dm'\right). \tag{11}$$

Some forms of $P_{KP}(m)$ are distinguished. The simplest choice is $P_{KP}(m)\sim\delta(m-\nu_0)+\delta(m+\nu_0)$ and corresponds to the dichotomous noise (the random telegraph process) [27]. Due to the central limit theorem, the Gaussian distribution of $P_{KP}(m)$ is of special importance. The Kubo-Anderson process with that distribution resembles the Ornstein-Uhlenbeck process. However, both processes are not identical [28]; the Fokker-Planck equation for the Ornstein-Uhlenbeck process, $(\partial)/(\partial t)p(m,t)=\nu_0(\partial)/(\partial m)(m+D\nu_0(\partial)/(\partial m))p(m,t)$, differs from the Eq. (11).

B. Stretched exponential covariance

The exponential distribution of interval lengths $P(s)$, a distinctive feature of the Kubo-Anderson process, can also characterize stochastic processes with the covariance form other than exponential. Let us consider the covariance possessing the shape known as the ''stretched exponential.'' This function is the following:

$$C(t)=\exp(-\alpha t^{\gamma}), \tag{12}$$

where $\alpha=$const and $0<\gamma<1$. In the present study, we assume $\gamma=1/2$. The stretched exponential function describes relaxation phenomena in random systems [29,30] and can be attributed to a dispersive transport of mobile defects in the glass [31]. The dispersive transport is characterized by the infinite average time between subsequent hops. Moreover, the velocity autocorrelation function of a particle inside the Sinai billiard with the finite horizon is also given by the Eq. (12) [32].

We want to find expressions, useful in practical applications, allowing us to generate time series possessing the required covariance. The inverted Laplace transform of Eq. (12) reads

$$\tilde{C}(\nu)=\frac{1}{2}\frac{\alpha}{\sqrt{\pi}}\nu^{-3/2}\exp[-\alpha^2/(4\nu)]. \tag{13}$$

In order to simplify the differential equation (9), we take the amplitude distribution in the form

$$P_{KP}(m)=\frac{2}{\pi}|m|^{-5}\exp(-\pi^{-1}m^{-4}), \tag{14}$$

where $m\in(-\infty,\infty)$. The distribution possesses maxima at $m=m_{max}=\pm[4/(5\pi)]^{1/4}$ and it is very small near $m=0$. Inserting Eqs. (13) and (14) into Eq. (9), we get the differential equation for $\nu(m)$ in the form

$$\frac{d\nu}{dm}=8/(\sqrt{\pi}\alpha)m^{-3}\nu^{3/2}\exp\left[-\frac{\alpha^2}{4}\{\nu^{-1}-4/(\alpha^2\pi m^4)\}\right]. \tag{15}$$

One can easily check that Eq. (15) is satisfied by the function

$$\nu(m)=1/s=(\pi\alpha^2/4)m^4 \tag{16}$$

and the interval lengths distribution is indeed exponential:

$$P(s)=\frac{\alpha^2}{4}\exp\left(-\frac{\alpha^2}{4}s\right). \tag{17}$$

The direct relation between s and m follows from the Eq. (16). From that equation, we conclude that $s\in(0,\infty)$ and long intervals correspond to the values of m close to zero; such events are extremely rare. Technically, a time series $m(t)$ can be constructed by sampling subsequent intervals s from the distribution $P(s)$. Then corresponding process values are evaluated by means of Eq. (16), taking into account, in addition, that both signs of m are equally probable.

C. Algebraic covariance

Let us now consider the KP possessing the power-law covariance that we express in the following form:

$$C(t)=\Gamma(\gamma)t^{-\gamma}\quad(\gamma>0). \tag{18}$$

The jumping frequency $\nu(m)$ one can derive from Eq. (9), similarly as for the stretched exponential case

$$\nu(m)=\left[2\gamma\int_0^{|m|}m'^2P_{KP}(m')dm'\right]^{1/\gamma}. \tag{19}$$

We assume the amplitude distribution in the algebraic form: $P_{KP}(m)\sim m^{-\alpha}$ ($\alpha=$const), modified in order to satisfy the normalization condition $2\int_0^\infty P_{KP}(m)dm=1$. Generally, two different forms of $P_{KP}(m)$ are possible.

In the first case, we cut off the large values of $|m|$

$$P_{KP}(m)=\begin{cases}\dfrac{1-\alpha}{2a}(|m|/a)^{-\alpha} & |m|\leqslant a\\ 0 & |m|>a,\end{cases} \tag{20}$$

where $a=$const is an additional parameter. Due to the condition $\nu(\infty)=\infty$, a must be a large number; finiteness of a results in a deviation of the covariance from the assumed form (18) near $t=0$ and in removal of the singularity. Moreover, the normalization condition implies $\alpha<1$. Inserting $P_{KP}(m)$ into the Eq. (19) gives us the expression for the relation between m and s,

$$s=\frac{1}{\nu}=(\gamma\alpha')^{-1/\gamma}a^{(1-\alpha)/\gamma}|m|^{-(3-\alpha)/\gamma}\theta(s-\epsilon), \tag{21}$$

where we have introduced a constant $\alpha'=(1-\alpha)/(3-\alpha)$. Finiteness of the parameter a imposes a restriction on the lower bound of the interval length ϵ: $s\in(\epsilon,\infty)$. The smallest interval length ϵ is related to that parameter by: $\epsilon=(\alpha'\gamma)^{-1/\gamma}a^{-2/\gamma}$. The probability distribution of interval lengths takes the algebraic form

$$P(s)=(\gamma\alpha')^{1-\alpha'}a^{-2\alpha'}s^{-\gamma\alpha'-1}. \tag{22}$$

From Eq. (22), some restrictions on possible asymptotic behavior of $P(s)$ follow. The slowest decay rate for large s occurs for α close to 1: $P(s)\sim s^{-1}$. On the other hand, the distribution $P(s)$ falls the most rapidly, as $\sim s^{-\gamma-1}$, for $\alpha\to-\infty$.

The other possibility is to cut off the distribution $P_{KP}(m)$ near $m=0$:

$$P_{KP}(m)=\begin{cases}0 & |m|<a\\ \dfrac{\alpha-1}{2a}(|m|/a)^{-\alpha} & |m|\geqslant a,\end{cases} \tag{23}$$

where $\alpha>1$. One can show that the minimal interval length is finite (nonzero) if $\alpha>3$. The interval lengths distribution can be obtained similarly as for the case (20). The result is slightly more complicated:

$$P(s)=\left(\frac{3-\alpha}{\gamma(\alpha-1)}a^{1-\alpha}s^{-\gamma}+a^{3-\alpha}\right)^{-2/(3-\alpha)}s^{-\gamma-1}. \tag{24}$$

Asymptotically, the distribution approaches $s^{-\gamma-1}$, independently of α.

III. APPLICATION TO THE GLE

We will now consider Eq. (1) for which the random force $F(t)$ is modeled by means of a concrete stochastic process, possessing a given covariance, and simulate stochastic trajectories of the Brownian particle by means of a Monte Carlo method. Therefore, two equations are to be solved simulta-

neously: the original GLE and the second one, describing an adjoined process, in the form of some KP. Accordingly, in the following, we assume $F(t)=m(t)$. A similar approach used to be applied [33,34] to deal with stochastic equations driven by the colored noise; the adjoined process constitutes in that case the Ornstein-Uhlenbeck process. Independently of those methods, which model the noise as a stochastic, random process, one can introduce some deterministic system possessing a required autocorrelation function. The value of the noise at a given time is then determined by the evolution of dynamical equations of motion. In this way, Shimizu [35] solved both ordinary Langevin equation and GLE, representing the noise by a chaotic map. Similarly, the Sinai billiard has been used to model a noise with long-time covariance in the Langevin equation [36]. Certainly, the first two moments do not determine the noise uniquely and the choice of a model involves additional assumptions, especially for non-Gaussian and not exponentially correlated processes. The form of the noise must be decided according to physical requirements of a concrete application. Nevertheless, some important quantities, like the autocorrelation function of the Brownian particle velocity (5), apparently depend only on the noise covariance.

However, for some stochastic processes description of those quantities provided by the equations collected in the Introduction does not agree with the simulation results. Solving the GLE to obtain the Brownian particle velocity requires the value of the noise $F(t)$, determined by the adjoined process, at a given time. It has been demonstrated recently [37] that this requirement modifies probability distributions–the adjoined process looks differently when inserted into the equation. Consequences of that change for the Monte Carlo simulation results may be important. In particular, one can expect a violation of the FDT that manifests itself in an improper asymptotic behavior of the velocity variance $\langle v^2(t)\rangle_S$. Determining of that quantity can indicate whether the expected equilibrium state is reached and the equipartition energy rule holds.

The Brownian particle velocity is to be determined from the stochastic integral (2) in which stepwise, constant values of $F(t)$, F_k follow from the length of subsequent time intervals s_k:

$$v(t)=m^{-1}\left[\sum_{k=1}^{n} F_k\int_{t_{k-1}}^{t_k} R(t-\tau)d\tau+F_{n+1}\int_{t_n}^{t} R(t-\tau)d\tau\right], \tag{25}$$

where $t_k-t_{k-1}=s_k$ and $t_0=0$. Sampling of intervals s_k continues as long as the time t is reached: $\Sigma_1^n s_k<t$ and $\Sigma_1^{n+1}s_k \geqslant t$, where n is an arbitrary integer. The interval length distribution $P(s)$ is a natural quantity of interest. The last $n+1$ interval corresponds to the time t [38]. It is clear that the distribution of lengths of that interval, the ''effective'' interval distribution $\hat{P}(s,t)$, cannot be identical with $P(s)$; a simple consideration reveals, e.g., the enhanced probability of choosing longer intervals. Generally, that modified distribution can depend on t. One can express it in terms of the cumulative distribution function $\Phi(s,t)$ [37]:

$$\hat{P}(s,t)=\frac{\partial}{\partial s}\Phi(s,t), \tag{26}$$

where

$$\Phi(s,t)=\begin{cases}\displaystyle\int_{t-s}^{t} S(x)dx\int_{t-x}^{s} P(\xi)d\xi & \text{for } 0\leqslant s\leqslant t\\ \displaystyle\int_{0}^{t} S(x)dx\int_{t-x}^{s} P(\xi)d\xi+\int_{t}^{s} P(\xi)d\xi & \text{for } s>t.\end{cases} \tag{27}$$

The form of auxiliary function $S(x)$ follows from the normalization condition

$$\int_0^t S(x)dx\int_{t-x}^{\infty} P(\xi)d\xi+\int_t^{\infty} P(\xi)d\xi=1. \tag{28}$$

For the Kubo-Anderson process, the modification of the interval distribution $P(s)$, exponential in that case, is of minor importance because $P_{KP}(F)$ is independent of $P(s)$ and the interval lengths do not influence process values. Consequently, results of the simulations agree with general predictions implied by the FDT. Other forms of noise covariance require taking into account the modified distribution $\hat{P}(s,t)$. The case of stretched exponential covariance provides a simple but nontrivial example. It involves also the Poissonian distribution of jumping times.

A. Stretched exponential noise covariance

We assume the noise covariance in the following form:

$$C_F(t)=2mT/\alpha^2\exp(-\alpha\sqrt{t}). \tag{29}$$

Equations (2) and (25) express the solution of GLE in terms of the resolvent $R(t)$. The Laplace transform of that function $\tilde{R}(s)$, is given by Eq. (3) and the kernel has the form

$$\tilde{K}(s)=\frac{\alpha^2}{2}\left(\frac{1}{s}-\frac{\sqrt{\pi}\alpha}{2s\sqrt{s}}\exp(\alpha^2/(4s))\left[1-\mathrm{erf}\left(\frac{\alpha}{2\sqrt{s}}\right)\right]\right), \tag{30}$$

where erf(x) denotes the error function. To obtain the resolvent $R(t)$ we need to evaluate the inverse Laplace transform from $\tilde{R}(s)$: $R(t)=(1/2\pi i)\int_{-i\infty+\sigma}^{+i\infty+\sigma}\tilde{R}(z)e^{tz}dz$. The integrand possesses two conjugate simple poles and a cut along the negative real axis. Evaluation of the contour integral produces the following result:

$$R(t)=e^{-at}(c_1\sin bt+c_2\cos bt)-\frac{4}{\sqrt{\pi}}\int_0^\infty\frac{x^2\exp[x^2-\alpha^2t/(4x^2)]dx}{[\{2x^2+\alpha^2/(4x^2)\}\exp(x^2)-2\sqrt{\pi}x^3\,\mathrm{erfi}(x)]^2+4\pi x^6}; \tag{31}$$

the imaginary error function erfi $(x)\equiv -i$ erf (ix) can easily be calculated by the following expansion:

$$\mathrm{erfi}(x)=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!(2n+1)}.$$

The constants a and b denote the real and imaginary parts of the pole of $\tilde{R}(z)$, respectively: $z_0=-a-|b|i$; they have to be evaluated numerically. c_1 and c_2 can be found by the standard residues analysis. For $\alpha=1$ the constants are the following: $a=0.207\,094$, $b=0.440\,963$, $c_1=-0.127\,752$, and $c_2=0.593\,952$. Figure 1 presents the function $R(t)$ for $\alpha=1$ and $\alpha=2$.

We wish to perform the Monte Carlo simulation using the noise defined by the process (14). According to Eq. (17), the interval distribution is exponential: $P(s)=\beta\exp(-\beta s)$, where $\beta=\alpha^2/4$. We expect that taking into account of the modified form of the distribution may be important for the simulation results because m and s are connected. That distribution can easily be found in this case. From Eq. (28), we obtain $S(x)=\beta$; finally we get

$$\hat{P}(s,t)=\begin{cases}\beta^2 s\,\exp(-\beta s) & \text{for}\quad 0\leqslant s\leqslant t\\ \beta(1+\beta t)\exp(-\beta s) & \text{for}\quad s>t.\end{cases} \tag{32}$$

The function (32) is presented in Fig. 2. The distribution is discontinued. It depends on t but this dependence dwindles exponentially with time; the left-hand branch $(s\leqslant t)$ is time independent. Therefore, asymptotically the process becomes stationary. Nevertheless, $\hat{P}(s,t)$ possesses a mean value twice that for $P(s)$ [39] and simulation results must reflect that. Indeed, the velocity variance obtained from the Eq. (25), shown in Fig. 3, stabilizes at a lower value then that predicted by the equipartition energy rule (T/m). It is so because long intervals correspond to small values of the noise amplitude, according to the Eq. (16). For short times, in turn, the result of the simulation agrees with the general prediction (4), also shown in Fig. 3, because then the branch $s>t$ dominates the distribution (32) and the dependence on t is weak.

It is possible to construct some KP that does not change when inserted into the GLE and that produces the equilibrium state in agreement with the FDT. For that purpose we single out some subset of kangaroo processes, a *restricted* KP (RKP), defined in the following way. We choose subsequent intervals s_k in the stepwise evolution of KP, according to the distribution $P(s)$, and assume that t corresponds to the $n+1$ interval, i.e., $S_n\equiv s_1+s_2+\cdots+s_n<t$ and $S_{n+1}>t$. Let $S_t\equiv S_{n+1}-t>0$. We call some KP "restricted" if $S_t\leqslant d$ for a given d; n is an arbitrary integer: $n\in(0,\infty)$. The probability distribution of the last interval lengths we denote by $P^\star(s,t;d)$. Obviously, $P^\star(s,t;\infty)=\hat{P}(s,t)$. On the other hand, in the limit $d\to 0$ the last interval lengths obey the

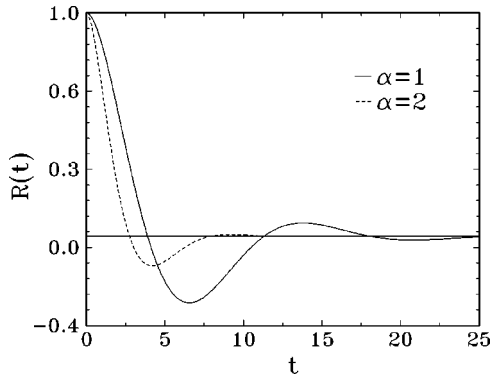

FIG. 1. The resolvent $R(t)$ (31) for the stretched exponential shape of noise covariance as a function of time.

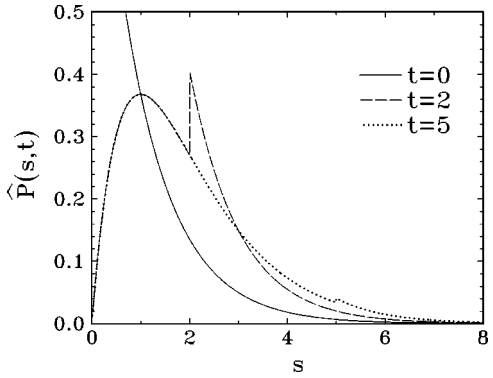

FIG. 2. The time evolution of the effective interval distribution $\hat{P}(s,t)$ (32) corresponding to the exponential form of the original distribution $P(s)$ with $\beta=1$.

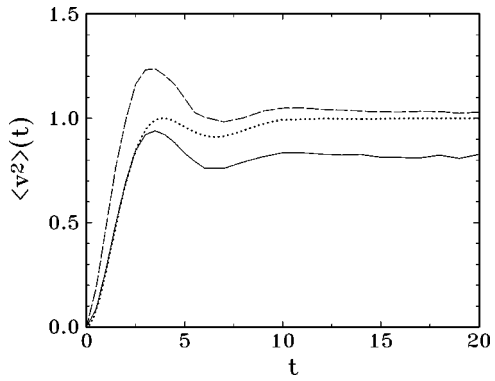

FIG. 3. The velocity variance obtained from the Monte Carlo simulation of GLE solutions (25) for stretched exponential form of noise covariance. The noise has been modeled by the KP (solid line) and the RKP with $d=0.01$, using theorem (33) (dashed line). 1000 trajectories have been calculated for either case. The variance calculated from Eq. (4) is also presented (dots). The temperature $T=1$, the particle mass $m=1$, and the parameter $\alpha=1$.

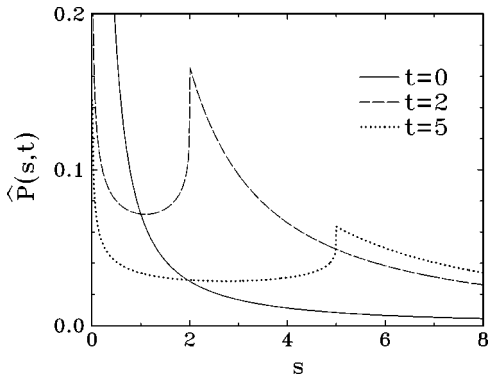

FIG. 4. The time evolution of the effective interval distribution $\hat{P}(s,t)$ (38) corresponding to the original distribution $P(s)$ in the form (35) with $\gamma=1$. The minimal interval length $\epsilon=0.01$.

original, time-independent distribution $P(s)$ but only for $s\leqslant t$ because longer intervals are excluded by construction. Therefore, the distribution $P(s)$ can be recovered at large times if those intervals are negligible. We get then the following theorem: If the probability that intervals in a sequence s_k are larger than t asymptotically vanishes, then

$$\lim_{t\to\infty}\lim_{d\to 0} P^{\star}(s,t;d)=P(s). \tag{33}$$

For the distribution (32), the required probability vanishes with time and the theorem can be applied. In practice it can be done easily by choosing some small d and sampling intervals from the distribution $P(s)$, in the same way as before, as long as the time t is reached. Then all sequences of intervals for which $S_t>d$ are rejected. Figure 3 presents the result of such calculations for $d=0.01$. At short times, the velocity variance differs substantially from the other results shown in the figure because the branch $s>t$ is then essential, but asymptotically it approaches the value T/m, in accordance with the equipartition energy rule.

B. Algebraic noise covariance

Finally, we consider a power-law form of the covariance

$$C_F(t)\sim t^{-\gamma} \quad (0<\gamma<3). \tag{34}$$

The KP we apply to model the noise is defined by Eq. (20); we implement the simplest case $\alpha=0$. Then the amplitude distribution is a constant (except very large $|F|$): $P_{KP}(F)=\sqrt{\gamma\epsilon^{\gamma}/3}/2$ where ϵ is the smallest interval length. The interval distribution follows from Eq. (22):

$$P(s)=\frac{\gamma}{3}\epsilon^{\gamma/3}s^{-1-\gamma/3}\theta(s-\epsilon). \tag{35}$$

The resolvent $R(t)$ for the noise with covariance (34) can be evaluated by means of similar methods as for the stretched exponential covariance. Results can be found in Ref. [40] for $\gamma=3/2$ and in Refs. [37,41] for $\gamma=1$. Now we want to calculate the modified distribution $\hat{P}(s,t)$. First, we have to solve Eq. (28), which assumes the form of Abel's integral equation

$$\int_0^t S(x)(t-x)^{-\gamma/3}dx+t^{-\gamma/3}=\epsilon^{-\gamma/3}. \tag{36}$$

The solution reads

$$S(x)=\frac{\epsilon^{-\gamma/3}}{\Gamma(1-\gamma/3)\Gamma(\gamma/3)}x^{\gamma/3-1}-\delta(x). \tag{37}$$

After evaluation of integrals, we obtain from Eqs. (26) and (27) the expression for the required distribution:

$$\hat{P}(s,t)=\begin{cases}\dfrac{s^{-\gamma/3-1}}{\Gamma(1-\gamma/3)\Gamma(\gamma/3)}[t^{\gamma/3}-(t-s)^{\gamma/3}] & \text{for } \epsilon\leqslant s\leqslant t\\ \left[\dfrac{t^{\gamma/3}}{\Gamma(1-\gamma/3)\Gamma(\gamma/3)}+\dfrac{\gamma\epsilon^{\gamma/3}}{3}\right]s^{-\gamma/3-1} & \text{for } s>t.\end{cases} \tag{38}$$

The distribution $\hat{P}(s,t)$ for $\gamma=1$ is presented in Fig. 4. The picture is markedly different from that obtained for the exponential case (Fig. 2); the right branch, corresponding to the intervals $s>t$, does not vanish with time but gets larger when comparing to the left branch. The entire distribution shifts with time towards large intervals because the average interval length is infinite.

Therefore, the effective distribution $\hat{P}(s,t)$ differs substantially from the $P(s)$, and the stochastic process generated by it must possess different properties. First, let us recalculate the covariance that, in general, can depend on an initial time t_0: $\hat{C}(t,t_0)=\langle F(t_0)F(t_0+t)\rangle_S$, where the process $F(t)$ is to be determined by the simulation. Technically, that means that for a given t_0, one produces a sequence of intervals to reach the time t_0+t. Then one evaluates corresponding values of the process $F(t)$ using Eq. (21). Then the pro-

cess is governed by the distribution $\hat{P}(s,t)$. The expression for the covariance follows from the Eq. (7):

$$\hat{C}(t,t_0)=\int_{\epsilon}^{\infty} s^{-2\gamma/3}\exp(-t/s)\hat{P}(s,t_0)ds. \tag{39}$$

Evaluation of the integral gives the following result:

$$\hat{C}(t,t_0)=\frac{3\epsilon^{-\gamma/3}\gamma^{-1/3}}{\Gamma(1-\gamma/3)\Gamma(\gamma/3)}[t_0^{\gamma/3}t^{-\gamma}\overline{\gamma}(\gamma,t/\epsilon) - t_0^{1/2-\gamma/6}t^{(-1-\gamma)/2}\exp(-t/2t_0) \times\Gamma(1+\gamma/3)W_{\gamma/6-1/2,-\gamma/2}(t/t_0)], \tag{40}$$

where $\overline{\gamma}(a,x)$ denotes the incomplete gamma function and $W_{\alpha,\beta}(x)$ stands for the Whittaker function [42]. Certainly, the above result is different from our starting covariance (34); the most striking feature of the function $\hat{C}(t,t_0)$ is the dependence on t_0 that does not diminish with t. The variance of the process, $\hat{\sigma}^2(t_0)$, can be found by inserting $t=0$ into the Eq. (40). Let us consider two examples. The case $\gamma=1$ has been discussed in Ref. [37]; the final expression for the variance is the following:

$$\hat{\sigma}^2(t_0)=\frac{\epsilon^{-1/3}}{\Gamma(1/3)\Gamma(2/3)}[3\ln 3/2+\pi\sqrt{3}/6+\ln(t_0/\epsilon)]t_0^{-2/3} \quad (t_0\gg\epsilon). \tag{41}$$

Algebraically correlated stochastic processes for $\gamma=3/2$ are especially important. They have been extensively studied in connection with the Brownian motion in a viscous fluid [43,4–6,40]. In this case, the variance of the process reads

$$\hat{\sigma}^2(t_0)=(\tfrac{2}{3})^{1/3}\left(\frac{2}{\pi\epsilon}t_0^{-1/2}+t_0^{-3/2}\right). \tag{42}$$

Therefore, in contrast to the original variance [calculated with the distribution $P(s)$] $\sigma^2=\gamma^{-1}\epsilon^{-\gamma}$, the effective variance $\hat{\sigma}^2$ is time dependent and tends to zero. That behavior is a direct consequence of nonstationarity, i.e., of the time dependence of the distribution $\hat{P}(s,t)$. The decline of $\hat{\sigma}^2(t_0)$ means that the effective temperature of the system drops to zero when we insert the process into the stochastic integral (25) as a model of the noise. Consequently, the Brownian particle velocity variance also must dwindle with time. Indeed, a direct Monte Carlo simulation confirms this conclusion, as it has been demonstrated in Ref. [37] for $\gamma=1$.

The RKP can be constructed also for algebraic correlations. However, the theorem (33) cannot be applied because of the strong time dependence of the distribution $\hat{P}(s,t)$. The probability that the interval length is larger then t does not decline with time: $\hat{P}(s>t,t)=\int_t^{\infty}\hat{P}(s,t)ds=3/[\gamma\Gamma(1-\gamma/3)\Gamma(\gamma/3)]=\text{const}$.

IV. SUMMARY AND DISCUSSION

The kangaroo processes represent a broad class of random functions characterized by various forms of the covariance. Therefore, they provide an opportunity to model physical stochastic processes possessing an arbitrary covariance and, in addition, a quite general amplitude distribution. The KP is stepwise–the value of the process changes according to some jumping frequency that, in turn, depends on that value. Some physical phenomena exhibit a similar, stepwise behavior and the KP is a natural process to model them. An important quantity is the distribution of intervals of constant process value $P(s)$, uniquely connected to the amplitude distribution. In this paper, we have discussed two forms of that distribution: the exponential and the algebraic ones. We have demonstrated how one can generate algebraically correlated processes using the KP with some algebraic $P(s)$. The exponential form, in turn, is suitable to represent both exponential and stretched exponential correlations.

The GLE has been solved using KP as a model of the noise. The problem has been considered as a juxtaposition of two random processes: of velocity of the Brownian particle, described by the GLE, and of the adjoined KP. In the framework of that approach, the force $F(t)$ in the stochastic integral (2) is determined by looking for a value of the KP, *independently* evolved, at a given time. Such procedure changes probability distributions of the KP and Monte Carlo simulated solutions of the GLE are not in agreement with results predicted by general analysis, founded on the FDT. Results obtained from effective probability distributions, among which the interval distribution $\hat{P}(s,t)$ is the most important, does not correspond to the equilibrium state consistent with the equipartition energy rule. For algebraically correlated processes, even nonstationarity effects emerge. Are those results involving effective distributions a necessary consequence of modeling of the noise by means of adjoined random process in the form of KP? If the stochastic force we want to insert into the stochastic integral (2) represents some independent physical process, the modified probability distribution (26) has to be taken into account. In such cases, we must expect an apparent violation of the FDT, despite proper definition of the kernel. Reversely, any information about properties of the noise, extracted from GLE solutions, always refers to the effective appearance of that process in the GLE.

On the other hand, for some cases, there is a possibility of constructing the model in such a way as to avoid any modification of the distributions and to preserve consistency with the FDT. For that purpose, one can use some specific version of the ordinary KP—the RKP. The idea is very simple: if some interval in the stepwise evolution of the KP ends exactly at a point corresponding to the required time, nothing has to be modified. We can imagine the KP as a ''clock'' with a variable frequency given by the distribution $P(s)$. In that picture, the application of the RKP with $d=0$ means a synchronization of that clock in respect to the physical time in the GLE. An important limitation of the synchronization procedure consists in the fact that interval lengths of the RKP are always finite, not longer than time t, and $P(s)$ usually

possesses infinite tails. If, however, those tails decline sufficiently fast, intervals longer then t become negligible. It has been demonstrated that the exponential interval distribution possesses this property and a Monte Carlo algorithm utilizing the RKP can easily be constructed—simulation results indeed correspond to the equilibrium state predicted by the FDT. The KP in this form is especially suitable for physical applications. We note, however, that the RKP with $d=0$ is not completely independent of the GLE: the synchronization introduces a coupling. The case of stretched exponential covariance is only the simplest nontrivial example of application of KP with the Poissonian interval lengths distribution. A straightforward generalization, allowing for the other power-law dependences $m(s)$, produces KP's with covariances given by the Bessel functions $\mathrm{K}_\nu(\sqrt{t})$.

One cannot expect that every shape of the covariance function may be modeled by some sufficiently steep form of $P(s)$ and, therefore, the RKP is always a proper tool. Processes possessing covariances with long tails are characterized by long intervals (long free paths). Indeed, in Sec. II, we have indicated a strong limitation of admissible shapes of $P(s)$. The algebraic tails of $P(s)$ are essential and very long intervals given by the effective distribution $\hat{P}(s,t)$ are not negligible for long times; the distribution itself remains time dependent. For that reason, the RKP that consist in a cutoff of long intervals cannot work as a model of algebraic covariances. In the other words, due to divergence of moments of both distributions, $P(s)$ and $\hat{P}(s,t)$, the branch corresponding to the intervals $s>t$ is important for large t and the synchronization cannot be achieved.

The necessity of taking into account the modified form of probability distributions is not restricted to the GLE but relates to any stochastic equation, e.g., the ordinary Langevin equation, if the random force is modeled by the adjoined KP. Monte Carlo simulations can be useful for some generalizations of the diffusion equation, in particular for the Burgers equation [44]. Nonlinear equations, possessing a broad spectrum of applications in the fluid dynamics, are characterized by long-range noise correlations both in space and time [45,1]. Also, noise can possibly be modeled by the KP.

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